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Parafermionic open string theory in noncommutative phase-space

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Abstract: We study how a free, open parafermionic string theory behaves when its target space is noncommutative. We consider noncommutativity in both space and momentum. We find new trilinear commutation relations for the string's oscillating modes and modified Virasoro superalgebras with additional anomaly terms. This noncommutativity breaks Lorentz invariance and makes the mass operator non-diagonal. To address these issues, we propose a new Fock space that diagonalizes the noncommutativity parameter matrices, leading to a diagonalized mass operator. We also impose constraints on the noncommutativity parameters to eliminate the anomalies and recover the usual mass spectrum. This allows for the GSO projection, which restores spacetime supersymmetry. Finally, we impose additional constraints on the zero modes of the noncommutativity parameters to recover Lorentz invariance. In general, our work provides a clearer picture of how noncommutative structures can be meaningfully included in string theory and suggests that even subtle deformations can carry significant consequences for the theory's symmetry and dynamics.
Keywords: *GSO projection, Lorentz algebra, Noncommutativity, Parafermionic strings, Virasoro super-algebra.*

1. Introduction

The idea of paraquantization began in 1950 by Wigner [1] who showed that the bilinear canonical commutation relations are a particular solution in order to satisfy the wave-particle duality. In 1953, Green [2] generalized the creation-annihilation operator algebra for bosons and fermions based on trilinear commutation relations. This paraquantization is parametrized by the order Q such that $Q = 1$ represents the usual canonical quantization. One can check as an application of the paraquantization in string theory done by Ardalan and Mansouri [3] in which they considered the center-of-mass coordinates obey the ordinary commutation relations, however, the oscillations are written in the paraquantum mode. Another application was investigated [4-6] where the string variables verify the trilinear commutation relations. As results for the two approaches, new possibility of a critical dimensions are obtained: $D = 2 + \frac{24}{Q}$ for the parabosonic string, $D = 2 + \frac{8}{Q}$ for paraspining string and $D = 3 + \frac{24}{Q}$ for parabosonic membrane.

A bosonic string in noncommutative space-time [7] can be generalized into the paraquantum case [8] indeed, it was found that the Virasoro algebra contains a new term of anomaly and the reconstruction of the Fock space is needed, in order to restore the photon state. For a parabosonic string between two parallel *Dp* and *Dq* brane, and by taking some restriction on the noncommutative parameters, the model will be free of tachyon. Finally, the closed parabosonic string has been investigated, where they demonstrate a reduction in the spectrum, specifically restoring the critical massless graviton state.

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For the Fermionic case Hamam and Belaloui [9] and Hammam and Belaloui [10] between two parallel *Dp*-, *Dq*-branes in Ramond and Neveu–Schwarz sectors, the study examines the possible existence of a free tachyon model within this system under specific conditions related to paraquantization and brane dimensions. To validate the model, the partition function is calculated and compared to the results of degeneracies, showing a perfect match. This consistency also confirms the integrity of the Virasoro superalgebra.

The purpose of this paper, is to study the paraquantum extension of a free fermionic string propagating in a noncommutative target phase-space [11] we will define the paraquantization generalization. Then, we calculate the Virasoro para-super-algebra, evaluate the mass spectrum and examine the Lorentz invariance. Finally, we summarize our work and conclude.

2. Open Fermionic Strings

We begin by considering the dynamics of free fermionic strings propagating within a noncommutative target space [7, 8, 11-15]. The action governing these strings is described by:

$$S=-\frac{1}{2\pi}\int d\sigma d\tau\left\{\partial_\alpha X^\mu(\sigma,\tau)\partial^\alpha X_\mu(\sigma,\tau)-i\bar{\psi}^\mu(\sigma,\tau)\rho^\alpha\partial_\alpha\psi_\mu(\sigma,\tau)\right\} \tag{1}$$

where the noncommutation relations are given by:

$$\begin{aligned}
\left[X^\mu(\tau,\sigma),P^\nu(\tau,\sigma')\right]&=i\eta^{\mu\nu}\delta(\sigma-\sigma')\\
\left[X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')\right]&=i\theta^{\mu\nu}\delta(\sigma-\sigma')\\
\left[P^\mu(\tau,\sigma),P^\nu(\tau,\sigma')\right]&=i\gamma^{\mu\nu}\delta(\sigma-\sigma')\\
\left\{\psi^\mu(\tau,\sigma),\psi^\nu(\tau,\sigma')\right\}&=\eta^{\mu\nu}\delta(\sigma-\sigma')
\end{aligned} \tag{2}$$

and where $P^\mu(\tau,\sigma)=\frac{1}{2\pi\alpha'}\partial_\tau X^\mu(\tau,\sigma)$, $\theta^{\mu\iota}$ represent the noncommutativity parameters of the space part and $\gamma^{\mu\iota}$ the ones of the momentum part of the phase-space.

One can write the Fourier expansions for the variables $\theta^{\mu\nu}(\sigma-\sigma'),\gamma^{\mu\nu}(\sigma-\sigma')$ [7] and $X^\mu(\tau,\sigma),\psi^\mu(\tau,\sigma)$ [16-21]:

$$\theta^{\mu\nu}(\sigma-\sigma')=\sum_{n=-\infty}^{+\infty}\theta_n^{\mu\nu}e^{in(\sigma-\sigma')} \tag{3}$$

$$\gamma^{\mu\nu}(\sigma-\sigma')=\sum_{n=-\infty}^{+\infty}\gamma_n^{\mu\nu}e^{in(\sigma-\sigma')} \tag{4}$$

$$X^\mu(\tau,\sigma)=x^\mu+2\alpha' p^\mu\tau+i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\cos(n\sigma)e^{-in\tau} \tag{5}$$

$$NS-\mathrm{sec}tor:\psi^\mu(\tau,\sigma)=\frac{1}{\sqrt{2}}\sum_{r\in Z+\frac{1}{2}}b_r^\mu e^{-ir(\tau-\sigma)} \tag{6}$$

$$R-\mathrm{sec}tor:\psi^\mu(\tau,\sigma)=\frac{1}{\sqrt{2}}\sum_{n\in Z}d_n^\mu e^{-in(\tau-\sigma)}$$

Using (3), (4), (5) and (6), we can verify that the equations (2) are equivalent to the following modified commutation relations of the oscillator algebra [7]:

$$\begin{aligned}
\left[p^{\mu}, p^{\nu}\right] &= i\pi^2 \gamma_0{}^{\mu\nu} \\
\left[x^{\mu}, p^{\nu}\right] &= i\eta^{\mu\nu} - 2i\pi^2 \alpha' \tau \gamma_0{}^{\mu\nu} \\
\left[x^{\mu}, x^{\nu}\right] &= i\theta_0{}^{\mu\nu} - 4i\pi^2 \alpha'^2 \tau^2 \gamma_0{}^{\mu\nu}
\end{aligned} \tag{7}$$

$$\left[\alpha_m^{\mu}, \alpha_n^{\nu}\right] = \left(m\eta^{\mu\nu} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n{}^{\mu\nu} + i\frac{n^2}{2\alpha'}\theta_n{}^{\mu\nu} \right)\delta_{n+m,0} \tag{8}$$

$$\begin{cases} \left\{d_m^{\mu}, d_n^{\nu}\right\} = \eta^{\mu\nu}\delta_{m+n,0} \\ \left\{b_r^{\mu}, b_s^{\nu}\right\} = \eta^{\mu\nu}\delta_{r+s,0} \end{cases} \tag{9}$$

These modifications will impact the Virasoro super-algebra for both the Ramond and Neveu-Schwarz sectors, introducing new anomalies terms

3. Modified Virasoro Super-Algebra

The Virasoro generators in a quantized system are given by:
For Ramond sector:

$$L_m = L_m^{\alpha} + L_m^{d} = \begin{cases} L_m^{\alpha} = \frac{1}{2}\sum_{n\in Z} :\alpha_{-n}\alpha_{m+n}: \\ L_m^{d} = \frac{1}{2}\sum_{n\in Z}\left(n + \frac{1}{2}m\right) :d_{-n}d_{m+n}: \end{cases} \tag{10}$$

$$F_m = \sum_{n\in Z} \alpha_{-n} d_{m+n} \tag{11}$$

which represent the fermionic sector.
For Neuveu-Schwarz sector:

$$L_m = L_m^{\alpha} + L_m^{b} = \begin{cases} L_m^{\alpha} = \frac{1}{2}\sum_{n\in Z} :\alpha_{-n}\alpha_{m+n}: \\ L_m^{b} = \frac{1}{2}\sum_{r\in Z+\frac{1}{2}}\left(r + \frac{1}{2}m\right) :b_{-r}b_{m+r}: \end{cases} \tag{12}$$

$$G_r = \sum_{n\in Z} \alpha_{-n} b_{r+n} \tag{13}$$

which represent the bosonic sector.

Given the modifications to the oscillator algebra in (8), one can derive the modified Virasoro super-algebras for both sectors [11].

$$[L_m^{(\alpha)}, L_n^{(\alpha)}] = (m-n)L_{n+m}^{(\alpha)} + \frac{d}{12} m\left(m^2-1\right)\delta_{m+n,0} + R_{mn} \qquad (14)$$

where R_{mn} represent the anomaly part due to the noncommutativity, defined by:

$$R_{mn} = -\frac{1}{2}\sum_{p=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{p-n}^{\nu\mu} + \gamma_{m-p}^{\mu\nu}\right) + \frac{i}{2\alpha'}\left(\left(p-n\right)^2\theta_{p-n}^{\nu\mu} + \left(m-p\right)^2\theta_{m-p}^{\mu\nu}\right)\right]\alpha_p^\mu\alpha_{m+n-p}^\nu \qquad (15)$$

The super-algebra then, is:
For Neuveu-Schwarz sector:

8 $$[L_m, L_n] = (m-n)L_{n+m} + \frac{D}{8} m\left(m^2-1\right)\delta_{m+n,0} + R_{mn} \qquad (16)$$

$$\left[L_m, G_r\right] = \left(\frac{1}{2}m - r\right)G_{m+r} + V_{mr} \qquad (17)$$

$$\left\{G_r, G_s\right\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s} + B_{rs} \qquad (18)$$

where B_{rs}, V_{mr} are given by:

$$B_{rs} = -\frac{1}{2}\sum_{q=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{q-s}^{\nu\mu} + \gamma_{r-q}^{\mu\nu}\right) + \frac{i}{2\alpha'}\left(\left(q-s\right)^2\theta_{q-s}^{\nu\mu} + \left(r-q\right)^2\theta_{r-q}^{\mu\nu}\right)\right]b_q^\mu b_{r+s-q}^\nu \qquad (19)$$

$$V_{mr} = -\frac{1}{2}\sum_{q=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{q-r}^{\nu\mu} + \gamma_{m-q}^{\mu\nu}\right) + \frac{i}{2\alpha'}\left(\left(q-r\right)^2\theta_{q-r}^{\nu\mu} + \left(m-q\right)^2\theta_{m-q}^{\mu\nu}\right)\right]\alpha_q^\mu b_{r+m-q}^\nu \qquad (20)$$

For Ramond sector:

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{D}{8} m^3\delta_{m+n,0} + R_{mn} \qquad (21)$$

$$\left[L_m, F_n\right] = \left(\frac{1}{2}m - n\right)F_{m+n} + W_{mn}$$

$$\left\{F_r, F_s\right\} = 2L_{r+s} + \frac{D}{2} r^2\delta_{r+s} + D_{rs} \qquad (23)$$

with again D_{rs}, W_{mn} are given by:

$$D_{rs} = -\frac{1}{2}\sum_{q=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{q-s}^{\nu\mu} + \gamma_{r-q}^{\mu\nu}\right) + \frac{i}{2\alpha'}\left(\left(q-s\right)^2\theta_{q-s}^{\nu\mu} + \left(r-q\right)^2\theta_{r-q}^{\mu\nu}\right)\right]d_q^\mu d_{r+s-q}^\nu \qquad (24)$$

$$W_{mn} = -\frac{1}{2}\sum_{q=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{q-n}^{\nu\mu} + \gamma_{m-q}^{\mu\nu}\right) + \frac{i}{2\alpha'}\left(\left(q-n\right)^2\theta_{q-n}^{\nu\mu} + \left(m-q\right)^2\theta_{m-q}^{\mu\nu}\right)\right]\alpha_q^\mu d_{n+m-q}^\nu \qquad (25)$$

4. Modified Lorentz Algebra

The angular momentum $M^{\mu\nu}$ is given by:

$$M^{\mu\nu}=\begin{cases} x^{\mu}p^{\nu}-x^{\nu}p^{\mu}-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^{\mu}\alpha_{n}^{\nu}-\alpha_{-n}^{\nu}\alpha_{n}^{\mu}\right)- \\ \frac{i}{4}\sum_{r=-\infty}^{+\infty}\left(b_{-r}^{\mu}b_{r}^{\nu}-b_{-r}^{\nu}b_{r}^{\mu}\right)\rightarrow \text{N-S sector} \\ x^{\mu}p^{\nu}-x^{\nu}p^{\mu}-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^{\mu}\alpha_{n}^{\nu}-\alpha_{-n}^{\nu}\alpha_{n}^{\mu}\right)- \\ \frac{i}{4}\sum_{m=-\infty}^{+\infty}\left(d_{-m}^{\mu}d_{m}^{\nu}-d_{-m}^{\nu}d_{m}^{\mu}\right)\rightarrow \text{R-sector} \end{cases} \tag{26}$$

Using (7) (8) (9), a direct calculation leads to the following modified Lorentz algebra [11]:

$$\begin{aligned}[M^{\mu\nu},M^{\rho\lambda}]&=-i\eta^{\nu\rho}M^{\mu\lambda}+i\eta^{\mu\lambda}M^{\rho\nu}+\\ &\quad i\eta^{\nu\lambda}M^{\mu\rho}-i\eta^{\mu\rho}M^{\lambda\nu}+T^{\mu\nu\rho\lambda}\end{aligned} \tag{27}$$

$$[p^{\mu},M^{\nu\rho}]=i\eta^{\rho\mu}p^{\nu}-i\eta^{\nu\mu}p^{\rho}+K^{\nu\mu\rho} \tag{28}$$

$$\left[p^{\mu},p^{\nu}\right]=i\pi^{2}\gamma_{0}{}^{\mu\nu} \tag{29}$$

where $T^{\mu\nu\rho\lambda}$, $K^{\nu\mu\rho}$ represent the anomalies due to the noncommutativity and which are given by:

$$\begin{aligned}T^{\mu\nu\rho\lambda}&=i\pi^{2}\begin{pmatrix}\gamma_{0}^{\nu\lambda}x^{\mu}x^{\rho}+\gamma_{0}^{\nu\rho}x^{\mu}x^{\lambda}+\\ \gamma_{0}^{\mu\lambda}x^{\nu}x^{\rho}+\gamma_{0}^{\mu\rho}x^{\nu}x^{\lambda}\end{pmatrix}+\\ &2i\pi^{2}\alpha'\tau\begin{pmatrix}\gamma_{0}^{\nu\rho}x^{\mu}p^{\lambda}-\gamma_{0}^{\mu\lambda}x^{\rho}p^{\nu}+\\ \gamma_{0}^{\nu\lambda}x^{\mu}p^{\rho}-\gamma_{0}^{\mu\rho}x^{\lambda}p^{\nu}+\\ \gamma_{0}^{\mu\rho}x^{\nu}p^{\lambda}-\gamma_{0}^{\nu\lambda}x^{\rho}p^{\mu}+\\ \gamma_{0}^{\mu\lambda}x^{\nu}p^{\rho}-\gamma_{0}^{\nu\rho}x^{\lambda}p^{\mu}\end{pmatrix}+\\ &\left(i\theta_{0}^{\mu\rho}-4i\pi^{2}\alpha'^{2}\tau^{2}\gamma_{0}^{\mu\rho}\right)p^{\lambda}p^{\nu}+\\ &\left(i\theta_{0}^{\mu\lambda}-4i\pi^{2}\alpha'^{2}\tau^{2}\gamma_{0}^{\mu\lambda}\right)p^{\rho}p^{\nu}+\\ &\left(i\theta_{0}^{\nu\rho}-4i\pi^{2}\alpha'^{2}\tau^{2}\gamma_{0}^{\nu\rho}\right)p^{\lambda}p^{\mu}+\\ &\left(i\theta_{0}^{\nu\lambda}-4i\pi^{2}\alpha'^{2}\tau^{2}\gamma_{0}^{\nu\lambda}\right)p^{\rho}p^{\mu}+\\ &\left(i\frac{(2\pi\alpha')^{2}}{2\alpha'}\gamma_{n}{}^{\nu\rho}+i\frac{n^{2}}{2\alpha'}\theta_{n}{}^{\nu\rho}\right)\left(\alpha_{-n}^{\mu}\alpha_{n}^{\lambda}+\alpha_{-n}^{\lambda}\alpha_{n}^{\mu}\right)+\\ &\left(i\frac{(2\pi\alpha')^{2}}{2\alpha'}\gamma_{n}{}^{\mu\lambda}+i\frac{n^{2}}{2\alpha'}\theta_{n}{}^{\mu\lambda}\right)\left(\alpha_{-n}^{\rho}\alpha_{n}^{\lambda}+\alpha_{-n}^{\lambda}\alpha_{n}^{\rho}\right)+\\ &\left(i\frac{(2\pi\alpha')^{2}}{2\alpha'}\gamma_{n}{}^{\nu\lambda}+i\frac{n^{2}}{2\alpha'}\theta_{n}{}^{\nu\lambda}\right)\left(\alpha_{-n}^{\rho}\alpha_{n}^{\mu}+\alpha_{-n}^{\mu}\alpha_{n}^{\rho}\right)+\\ &\left(i\frac{(2\pi\alpha')^{2}}{2\alpha'}\gamma_{n}{}^{\mu\rho}+i\frac{n^{2}}{2\alpha'}\theta_{n}{}^{\mu\rho}\right)\left(\alpha_{-n}^{\lambda}\alpha_{n}^{\nu}+\alpha_{-n}^{\nu}\alpha_{n}^{\lambda}\right)\end{aligned} \tag{30}$$

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$$K^{\nu\mu\rho} = 2i\pi^2\alpha'\tau\gamma_0^{\nu\mu} p^\rho - 2i\pi^2\alpha'\tau\gamma_0^{\rho\mu} p^\nu + i\pi^2\gamma_0^{\mu\rho} p^\nu - i\pi^2\gamma_0^{\mu\nu} p^\rho \tag{31}$$

5. Paraquantization Generalization

Now, we are going to generalize the theory into the para-quantization, where we have a trilinear commutation relation, instead of the usual commutations. The generalization will be done in the light cone coordinates. By defining the Green's ansatz [6, 8-10] one can present the Green representation of the bosonic and fermionic string coordinates:

$$\begin{aligned} X^i(\tau,\sigma) &= \sum_{\alpha=1}^{Q} X^{(\alpha)i}(\tau,\sigma) \\ \psi_A^j(\tau,\sigma) &= \sum_{\alpha=1}^{Q} \psi_A^{(\alpha)j}(\tau,\sigma) \end{aligned} \tag{32}$$

Where $i = 2..D\!-\!2$ are the light cone coordinates, $\alpha = 1,2...Q$ which represent the Green indices, and Q represent the paraquantization order (taking $Q = 1$ we find results of the ordinary case). So, the generalization of canonical variables for the parabosonic and parafermionic coordinates are given by:

$$\begin{aligned}
&[X^{(\alpha)i}(\tau,\sigma), X^{(\alpha)j}(\tau,\sigma')] = i\theta^{ij}(\sigma-\sigma') \\
&\left[X^{(\alpha)i}(\tau,\sigma), X^{(\beta)j}(\tau,\sigma')\right]_+ = 0; \quad \alpha \neq \beta \\
&\left[P^{(\alpha)i}(\tau,\sigma), P^{(\alpha)j}(\tau,\sigma')\right] = i\gamma^{ij}(\sigma-\sigma') \\
&\left[P^{(\alpha)i}(\tau,\sigma), P^{(\beta)j}(\tau,\sigma')\right]_+ = 0; \quad \alpha \neq \beta \\
&\left[\psi_A^{(\alpha)i}(\tau,\sigma), \psi_B^{(\alpha)j}(\tau,\sigma')\right]_+ = \eta^{ij}\delta_{AB}\delta(\sigma-\sigma') \\
&\left[\psi_A^{(\alpha)i}(\tau,\sigma), \psi_B^{(\beta)j}(\tau,\sigma')\right]_- = 0; \quad \alpha \neq \beta \\
&\left[x^{(\alpha)-}, p^{(\alpha)+}\right] = i \\
&\left[x^{(\alpha)-}, p^{(\beta)+}\right]_+ = 0; \quad \alpha \neq \beta
\end{aligned} \tag{33}$$

The equations (33) are equivalent to the following tri-linear commutation relations:

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$$
\begin{aligned}
&\left[X^i(\tau,\sigma),\left[X^j(\tau,\sigma'),X^k(\tau,\sigma'')\right]_+\right]=2i\left\{\theta^{ij}(\sigma-\sigma')X^k(\tau,\sigma'')+\theta^{ik}(\sigma-\sigma'')X^j(\tau,\sigma')\right\}\\
&\left[X^i(\tau,\sigma),\left[P^j(\tau,\sigma'),X^k(\tau,\sigma'')\right]_+\right]=2i\left\{\eta^{ij}\delta(\sigma-\sigma')X^k(\tau,\sigma'')+\theta^{ik}(\sigma-\sigma'')P^j(\tau,\sigma')\right\}\\
&\left[X^i(\tau,\sigma),\left[P^j(\tau,\sigma'),P^k(\tau,\sigma'')\right]_+\right]=2i\left\{\eta^{ij}\delta(\sigma-\sigma')P^k(\tau,\sigma'')+\eta^{ik}\delta(\sigma-\sigma'')P^j(\tau,\sigma')\right\}\\
&\left[P^i(\tau,\sigma),\left[X^j(\tau,\sigma'),P^k(\tau,\sigma'')\right]_+\right]=2i\left\{-\eta^{ij}\delta(\sigma-\sigma')P^k(\tau,\sigma'')+\gamma^{ik}(\sigma-\sigma'')X^j(\tau,\sigma')\right\}\\
&\left[P^i(\tau,\sigma),\left[P^j(\tau,\sigma'),P^k(\tau,\sigma'')\right]_+\right]=2i\left\{\gamma^{ij}(\sigma-\sigma')P^k(\tau,\sigma'')+\gamma^{ik}(\sigma-\sigma'')P^j(\tau,\sigma')\right\}\\
&\left[X^i(\tau,\sigma),\left[\psi_A^j(\tau,\sigma'),X^k(\tau,\sigma'')\right]_+\right]=2i\theta^{ik}(\sigma-\sigma'')\psi_A^j(\tau,\sigma')\\
&\left[P^i(\tau,\sigma),\left[\psi_A^j(\tau,\sigma'),P^k(\tau,\sigma'')\right]_+\right]=2i\gamma^{ik}(\sigma-\sigma'')\psi_A^j(\tau,\sigma')\\
&\left[X^i(\tau,\sigma),\left[\psi_A^j(\tau,\sigma'),P^k(\tau,\sigma'')\right]_+\right]=2i\eta^{ik}\psi_A^j(\tau,\sigma')\delta(\sigma-\sigma'')\\
&\left[\psi_A^i(\tau,\sigma),\left[\psi_B^j(\tau,\sigma'),\psi_C^k(\tau,\sigma'')\right]_-\right]=2\left\{\eta^{ij}\delta_{AB}\psi_A^k(\tau,\sigma'')\delta(\sigma-\sigma')-\eta^{ik}\delta_{AC}\psi_B^j(\tau,\sigma')\delta(\sigma-\sigma'')\right\}\\
&\left[\psi_A^i(\tau,\sigma),\left[X^j(\tau,\sigma'),\psi_C^k(\tau,\sigma'')\right]_+\right]=2\eta^{ik}\delta_{AC}X^j(\tau,\sigma')\delta(\sigma-\sigma'')\\
&\left[\psi_A^i(\tau,\sigma),\left[P^j(\tau,\sigma'),\psi_C^k(\tau,\sigma'')\right]_+\right]=2\eta^{ik}\delta_{AC}P^j(\tau,\sigma')\delta(\sigma-\sigma'')\\
&\left[X^i(\tau,\sigma),\left[X^j(\tau,\sigma'),A\right]_+\right]=2i\theta^{ij}(\sigma-\sigma')A\\
&\left[X^i(\tau,\sigma),\left[P^j(\tau,\sigma'),A\right]_+\right]=2i\eta^{ij}(\sigma-\sigma')A\\
&\left[P^i(\tau,\sigma),\left[P^j(\tau,\sigma'),A\right]_+\right]=2i\gamma^{ij}(\sigma-\sigma')A\\
&\left[x^-,\left[p^+,B\right]_+\right]=2iB\\
&\left[x^-,\left[p^+,p^+\right]_+\right]=4ip^+
\end{aligned}
\tag{34}
$$

Now, in terms of modes, the trilinear equations take this form:

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$$\left[\alpha_m^i,\left[\alpha_n^j,\alpha_l^k\right]_+\right]=2\left\{\left(m\eta^{ij}+i\frac{n^2}{2\alpha'}\theta_n^{ij}+i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{ij}\right)\delta_{m+n}\alpha_l^k+\left(m\eta^{ik}+i\frac{n^2}{2\alpha'}\theta_l^{ik}+i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_l^{ik}\right)\delta_{m+l}\alpha_n^j\right\}$$

$$\left[\alpha_m^i,\left[\alpha_n^j,C\right]_+\right]=2\left\{\left(m\eta^{ij}+i\frac{n^2}{2\alpha'}\theta_n^{ij}+i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{ij}\right)\delta_{m+n}C\right\}$$

$$\left[x^-,\left[p^+,\alpha_m^i\right]_+\right]=2i\alpha_m^i$$

$$\left[p^i,\left[p^j,p^k\right]_+\right]=2i\frac{(2\pi\alpha')^2}{2\alpha'}\left\{\gamma_0^{ij}\delta_{m+n}p^k+\gamma_0^{ik}\delta_{m+n}p^j\right\}$$

$$\left[x^i,\left[p^j,p^k\right]_+\right]=2i\left\{\left(\frac{1}{2}\eta^{ij}-\frac{(2\pi\alpha')^2}{2\alpha'}\tau\gamma_0^{ij}\right)p^k+\left(\frac{1}{2}\eta^{ik}-\frac{(2\pi\alpha')^2}{2\alpha'}\tau\gamma_0^{ik}\right)p^j\right\}$$

$$\left[p^i,\left[x^j,x^k\right]_+\right]=-2i\left\{\left(\frac{1}{2}\eta^{ij}+\frac{(2\pi\alpha')^2}{2\alpha'}\tau\gamma_0^{ij}\right)x^k+\left(\frac{1}{2}\eta^{ik}+\frac{(2\pi\alpha')^2}{2\alpha'}\tau\gamma_0^{ik}\right)x^j\right\}$$

$$\left[x^i,\left[x^j,x^k\right]_+\right]=2i\left\{\left(\theta_0^{ij}-(2\pi\alpha')^2\tau^2\gamma_0^{ij}\right)x^k+\left(\theta_0^{ik}-(2\pi\alpha')^2\tau^2\gamma_0^{ik}\right)x^j\right\}$$

$$\left[d_n^i,\left[d_m^j,d_l^k\right]_-\right]=2\left(\delta^{ij}\delta_{n+m}d_l^k-\delta^{ik}\delta_{n+l}d_m^j\right) \quad (35)$$

$$\left[b_n^i,\left[b_m^j,b_l^k\right]_-\right]=2\left(\delta^{ij}\delta_{n+m}b_l^k-\delta^{ik}\delta_{n+l}b_m^j\right)$$

6. Virasoro Para-Super-Algebra

The Virasoro generators in the parafermionic strings are defined by:

$$L_m^{\perp}=\frac{1}{4}\sum_{i=2}^{D-1}\sum_{p=-\infty}^{\infty}\left[\alpha_{m-p}^i,\alpha_{pi}\right]_+ \quad (36)$$

as a result, the equations (16) to (25) will take this form:
For Ramond sector, we have:

$$L_m=L_m^{\alpha}+L_m^{d}=\begin{cases}L_m^{\alpha}=\frac{1}{4}\sum_{i=2}^{D-2}\sum_{p=-\infty}^{\infty}\left[\alpha_{m-p}^i,\alpha_{pi}\right]_+\\ L_m^{d}=\frac{1}{2}\sum_{i=2}^{D-2}\sum_{n\in Z}\left(n+\frac{1}{2}m\right)[d_{-n}^i,d_{im+n}]_-\end{cases} \quad (37)$$

$$F_m=\sum_{n\in Z}[\alpha_{-n},d_{m+n}]_+ \quad (38)$$

which represent the parafermionic sector.
For Neuveu-Schwarz sector, we have:

$$L_m = L_m^\alpha + L_m^b = \begin{cases} L_m^\alpha = \frac{1}{2}\sum_{i=2}^{D-2}\sum_{p=-\infty}^{\infty}\left[\alpha_{m-p}^i, \alpha_{pi}\right]_+ \\ L_m^b = \frac{1}{2}\sum_{i=2}^{D-2}\sum_{r\in Z+\frac{1}{2}}\left(r+\frac{1}{2}m\right)[b_{-r}^i, b_{im+r}]_- \end{cases} \quad (39)$$

$$G_r = \sum_{n\in Z}[\alpha_{-n}, b_{r+n}]_+ \quad (40)$$

which represent the parabosonic sector.

By using the trilinear commutations relations (35) and by analogy with calculation done in (21), (22) and (23), one can find that: for Ramond sector:

$$\begin{aligned} \left[L_n, L_m\right] &= (n-m)L_{n+m} + Q\frac{D-1}{8}n^3\delta_{n+m,0} + R_{nm} \\ \left[F_n, F_m\right]_+ &= 2L_{n+m} + Q\frac{D-1}{2}n^2\delta_{n+m,0} + D_{rs} \\ \left[L_n, F_m\right] &= \left(\frac{1}{2}n - m\right)F_{n+m} + W_{mn} \end{aligned} \quad (41)$$

where R_{nm}, D_{mn} and W_{mn} are given by:

$$R_{mn} = -\frac{1}{2}\sum_{p=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{p-n}^{ij} + \gamma_{m-p}^{ij}\right) + \frac{i}{2\alpha'}\left((p-n)^2\theta_{p-n}^{ij} + (m-p)^2\theta_{m-p}^{ij}\right)\right][\alpha_p^i, \alpha_{m+n-p}^j]_+ \quad (42)$$

$$D_{rs} = \frac{1}{2}\sum_{q=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{q-s}^{\nu\mu} + \gamma_{r-q}^{ij}\right) + \frac{i}{2\alpha'}\left((q-s)^2\theta_{q-s}^{ij} + (r-q)^2\theta_{r-q}^{ij}\right)\right]\left[d_q^i, d_{r+s-q}^j\right]_- \quad (43)$$

$$W_{mn} = \frac{1}{2}\sum_{q=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{q-n}^{ij} + \gamma_{m-q}^{\mu\nu}\right) + \frac{i}{2\alpha'}\left((q-n)^2\theta_{q-n}^{ij} + (m-q)^2\theta_{m-q}^{ij}\right)\right]\left[\alpha_q^i, d_{n+m-q}^j\right]_+ \quad (44)$$

for Neuveu-Schwartz sector:

$$\begin{aligned} \left[L_n, L_m\right] &= (n-m)L_{n+m} + Q\frac{D-1}{8}n(n^2-1)\delta_{n+m,0} + R_{nm} \\ \left[G_n, G_m\right]_+ &= 2L_{n+m} + Q\frac{D-1}{2}(n^2-\frac{1}{4})\delta_{n+m,0} + B_{rs} \\ \left[L_n, G_m\right] &= \left(\frac{1}{2}n - m\right)F_{n+m} + V_{mr} \end{aligned} \quad (45)$$

where R_{nm} is given by (42), and B_{rs} and V_{mr} are given by:

$$B_{rs} = \frac{1}{2}\sum_{q=-\infty}^{+\infty}\left[2i\alpha'\pi^2\left(\gamma_{q-s}^{ij} + \gamma_{r-q}^{ij}\right) + \frac{i}{2\alpha'}\left((q-s)^2\theta_{q-s}^{ij} + (r-q)^2\theta_{r-q}^{ij}\right)\right]\left[b_q^i, b_{r+s-q}^j\right]_- \quad (46)$$

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$$V_{mr}=\frac{1}{2}\sum_{q=-\infty}^{+\infty}\left[2i\alpha'\pi^{2}\left(\gamma_{q-r}^{ij}+\gamma_{m-q}^{ij}\right)+\frac{i}{2\alpha'}\left(\left(q-r\right)^{2}\theta_{q-r}^{ij}+\left(m-q\right)^{2}\theta_{m-q}^{ij}\right)\right]\left[\alpha_{q}^{i},b_{r+m-q}^{j}\right]_{+} \tag{47}$$

7. Mass Spectrum and GSO Projection

The mass operator will be written in the paraquantized form as [9, 10]:
Ramond sector:

$$M_{R}^{2}=\frac{1}{2\alpha'}\left(\sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left[\alpha_{-n}^{i},\alpha_{n}^{i}\right]_{+}+\sum_{i=2}^{D-1}\sum_{r=1}^{\infty}r\left[d_{-r}^{i},d_{r}^{i}\right]_{-}\right) \tag{48}$$

$$M_{NS}^{2}=\frac{1}{2\alpha'}\left(\sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left[\alpha_{-n}^{i},\alpha_{n}^{i}\right]_{+}+\sum_{i=2}^{D-1}\sum_{r=\frac{1}{2}}^{\infty}r\left[b_{-r}^{i},b_{r}^{i}\right]_{-}-Q\left(\frac{D-2}{16}\right)\right) \tag{49}$$

Neuveu-Schwarz sector:
We note that $D-2=\frac{8}{Q}$ [3-5].

We need to diagonalize the antisymmetric matrices θ_m and γ_m by introducing the unitary matrix U_m such that:

$$\left(U_{m}^{-1}i\theta_{m}U_{m}\right)^{ij}=D_{m}^{ij}=\mu_{i}^{(m)}\delta^{ij} \tag{50}$$

and,

$$\left(U_{m}^{-1}i\gamma_{m}U_{m}\right)^{ij}=T_{m}^{ij}=\nu_{i}^{(m)}\delta^{ij} \tag{51}$$

With $[\theta_m, \gamma_m] = 0$.

This last can be obtained through a redefinition of the Fock space [8, 22] in order to get a diagonal mass in this new basis. The redefinition takes this form:

$$\begin{aligned}&\left(\frac{1}{h!}\left\langle\prod_{i=2}^{D-1}\prod_{m=1}^{\infty}\left\{\left(\alpha_{-m}\right)^{i}\right\}^{\lambda_{m,i}}\right\rangle_{+}\left\langle\prod_{j=2}^{D-1}\left(\prod_{r=\frac{1}{2},\frac{3}{2}\ldots}\left(b_{-r}^{j}\right)^{\rho_{r,j}} or \prod_{n=1,2..}\left(d_{-n}^{j}\right)^{\rho_{n,j}}\right)\right\rangle_{-}\right)\left|p^{+},\vec{p}^{T}\right\rangle\rightarrow\\&\left(\frac{1}{h!}\left\langle\prod_{i=2}^{D-1}\prod_{m=1}^{\infty}\left\{\left(U_{m}^{-1}\alpha_{-m}\right)^{i}\right\}^{\lambda_{m,i}}\right\rangle_{+}\left\langle\prod_{j=2}^{D-1}\left(\prod_{r=\frac{1}{2},\frac{3}{2}\ldots}\left(b_{-r}^{j}\right)^{\rho_{r,j}} or \prod_{n=1,2..}\left(d_{-n}^{j}\right)^{\rho_{n,j}}\right)\right\rangle_{-}\right)\left|p^{+},\vec{p}^{T}\right\rangle\end{aligned} \tag{52}$$

Where $\langle\ldots\rangle_{\pm}$ represent the symmetrized (anti-symmetrized) form of the bosonic (Fermionic) oscillators product, $h=\sum_{m,i}\lambda_{m,i}$ represent its different possible permutations of the oscillators and $\rho_{n,k}$ takes either zero or one.

In order to get an equivalent of a GSO projection, one can use the usual way to get the following steps in the table below (Table 1) and (Table 2).

The results of GSO projection for the two sectors are grouped in (Table 3). (See (appendix A) as an example of calculations of mass spectrum).

Table 1.
This table represents the mass spectrum in terms of redefined modes.

Level	N-S Sector	
	state	**Mass**
0	$\|0\rangle$	$-\frac{1}{2\alpha'}$
1	$b^i_{-\frac{1}{2}}\|0\rangle$	0
2	$\frac{1}{2!}\left[b^i_{-\frac{1}{2}}, b^j_{-\frac{1}{2}}\right]_-\|0\rangle$	$\frac{1}{2\alpha'}$
	$U_1^{-1}\alpha^i_{-1}\|0\rangle$	$\frac{1}{\alpha'}\left(\frac{1}{2}-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$
3	$\frac{1}{3!}< b^i_{-\frac{1}{2}}, b^j_{-\frac{1}{2}}, b^k_{-\frac{1}{2}} >_-\|0\rangle$ $b^i_{-\frac{3}{2}}\|0\rangle$	$\frac{1}{\alpha'}$ $\frac{1}{\alpha'}$
	$\frac{1}{2!}\left[U_1^{-1}\alpha^i_{-1}, b^j_{-\frac{1}{2}}\right]_+\|0\rangle$	$\frac{1}{\alpha'}\left(1-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$
4	$\frac{1}{4!}< b^i_{-\frac{1}{2}}, b^j_{-\frac{1}{2}}, b^k_{-\frac{1}{2}}, b^l_{-\frac{1}{2}} >_-\|0\rangle$ $\frac{1}{2!}\left[b^i_{-\frac{3}{2}}, b^j_{-\frac{1}{2}}\right]_-\|0\rangle$ $U_2^{-1}\alpha^i_{-2}\|0\rangle$ $\frac{1}{2!}\left[U_1^{-1}\alpha^j_{-1}, U_1^{-1}\alpha^k_{-1}\right]_+\|0\rangle$ $\frac{1}{3!}< U_1^{-1}\alpha^j_{-1}, b^k_{-\frac{1}{2}}, b^l_{-\frac{1}{2}} >_+\|0\rangle$	$\frac{3}{2\alpha'}$ $\frac{3}{2\alpha'}$ $\frac{1}{\alpha'}\left(\frac{3}{2}-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\nu_j^{(2)}\right)\right)$ $\frac{1}{\alpha'}\left(\frac{3}{2}-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}+\mu_k^{(1)}+(2\pi\alpha')^2\nu_k^{(1)}\right)\right)$ $\frac{1}{\alpha'}\left(\frac{3}{2}-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$

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	$\frac{1}{5!}<b^i_{-\frac{1}{2}},b^j_{-\frac{1}{2}},b^k_{-\frac{1}{2}},b^l_{-\frac{1}{2}},b^m_{-\frac{1}{2}}>_-\|0\rangle$	$\frac{2}{\alpha'}$
	$b^i_{-\frac{5}{2}}\|0\rangle$	$\frac{2}{\alpha'}$
	$\frac{1}{3!}<b^i_{-\frac{1}{2}},b^j_{-\frac{1}{2}},b^k_{-\frac{1}{2}}>_-$	$\frac{2}{\alpha'}$
5	$\frac{1}{2!}\left[U_2^{-1}\alpha^j_{-2},b^k_{-\frac{1}{2}}\right]_+\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\nu_j^{(2)}\right)\right)$
	$\frac{1}{3!}<U_1^{-1}\alpha^j_{-1},U_1^{-1}\alpha^k_{-1},b^l_{-\frac{1}{2}}>_+\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}+\mu_k^{(1)}+(2\pi\alpha')^2\nu_k^{(1)}\right)\right)$
	$\frac{1}{4!}<U_1^{-1}\alpha^j_{-1},b^k_{-\frac{1}{2}},b^l_{-\frac{1}{2}},b^m_{-\frac{1}{2}}>_+\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$
	$\frac{1}{2!}\left[U_1^{-1}\alpha^j_{-1},b^k_{-\frac{3}{2}}\right]_+\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$

Table 2.
This table represents the mass spectrum in terms of redefined modes.

Level	R-Sector	
	State	**Mass**
0	$\|0\rangle$	0
1	$d^j_{-1}\|0\rangle$	$\underline{1}\alpha'$
	$U_1^{-1}\alpha^j_{-1}\|0\rangle$	$\frac{1}{\alpha'}\left(1-\left(\frac{1}{2\alpha'}\mu_j^{(1)}+\frac{(2\pi\alpha')^2}{2\alpha'}\nu_j^{(1)}\right)\right)$
2	$d^j_{-2}\|0\rangle$	$\frac{2}{\alpha'}$
	$\frac{1}{2!}\left[d^j_{-1},d^k_{-1}\right]_-\|0\rangle$	$\frac{2}{\alpha'}$
	$U_2^{-1}\alpha^j_{-2}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\nu_j^{(2)}\right)\right)$
	$\frac{1}{2!}\left[U_1^{-1}\alpha^j_{-1},U_1^{-1}\alpha^k_{-1}\right]_+\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\begin{array}{c}\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}+\\ \mu_k^{(1)}+(2\pi\alpha')^2\nu_k^{(1)}\end{array}\right)\right)$
	$\frac{1}{2!}\left[U_1^{-1}\alpha^j_{-1},d^k_{-1}\right]_+\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$

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Table 3.
This table represents the GSO projection for the two sectors.

Level	N-S Sector		R-Sector	
	State	Masse	State	Masse
1	$b^{i}_{-\frac{1}{2}}\mid 0\rangle$	0	$\mid 0\rangle$	0
3	$\frac{1}{3!}<b^{i}_{-\frac{1}{2}},b^{j}_{-\frac{1}{2}},b^{k}_{-\frac{1}{2}}>_{-}\mid 0\rangle$	$\frac{1}{\alpha'}$	$d^{j}_{-1}\mid 0\rangle$	$\frac{1}{\alpha'}$
	$b^{i}_{-\frac{3}{2}}\mid 0\rangle$	$\frac{1}{\alpha'}$		$\frac{1}{\alpha'}\left(1-\left(\frac{1}{2\alpha'}\mu_j^{(1)}+\frac{(2\pi\alpha')^2}{2\alpha'}\nu_j^{(1)}\right)\right)$
	$\frac{1}{2!}\left[U_1^{-1}\alpha^{i}_{-1},b^{j}_{-\frac{1}{2}}\right]_{+}\mid 0\rangle$	$\frac{1}{\alpha'}\left(1-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$	$U_1^{-1}\alpha^{j}_{-1}\mid 0\rangle$	
5	$\frac{1}{5!}<b^{i}_{-\frac{1}{2}},b^{j}_{-\frac{1}{2}},b^{k}_{-\frac{1}{2}},b^{l}_{-\frac{1}{2}},b^{m}_{-\frac{1}{2}}>_{-}\mid 0\rangle$	$\frac{2}{\alpha'}$	$d^{j}_{-2}\mid 0\rangle$	$\frac{2}{\alpha'}$
	$b^{i}_{-\frac{5}{2}}\mid 0\rangle$	$\frac{2}{\alpha'}$		
	$\frac{1}{3!}<b^{i}_{-\frac{1}{2}},b^{j}_{-\frac{1}{2}},b^{k}_{-\frac{1}{2}}>_{-}$	$\frac{2}{\alpha'}$	$\frac{1}{2!}\left[d^{j}_{-1},d^{k}_{-1}\right]_{-}\mid 0\rangle$	$\frac{2}{\alpha'}$
	$\frac{1}{2!}\left[U_2^{-1}\alpha^{j}_{-2},b^{k}_{-\frac{1}{2}}\right]_{+}\mid 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\nu_j^{(2)}\right)\right)$	$U_2^{-1}\alpha^{j}_{-2}\mid 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\nu_j^{(2)}\right)\right)$
	$\frac{1}{3!}<U_1^{-1}\alpha^{j}_{-1},U_1^{-1}\alpha^{k}_{-1},b^{l}_{-\frac{1}{2}}>_{+}\mid 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}+\mu_k^{(1)}+(2\pi\alpha')^2\nu_k^{(1)}\right)\right)$		
	$\frac{1}{4!}<U_1^{-1}\alpha^{j}_{-1},b^{k}_{-\frac{1}{2}},b^{l}_{-\frac{1}{2}},b^{m}_{-\frac{1}{2}}>_{+}\mid 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$	$\frac{1}{2!}\left[U_1^{-1}\alpha^{j}_{-1},U_1^{-1}\alpha^{k}_{-1}\right]_{+}\mid 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\begin{matrix}\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}+\\ \mu_k^{(1)}+(2\pi\alpha')^2\nu_k^{(1)}\end{matrix}\right)\right)$
	$\frac{1}{2!}\left[U_1^{-1}\alpha^{j}_{-1},b^{k}_{-\frac{3}{2}}\right]_{+}\mid 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$	$\frac{1}{2!}\left[U_1^{-1}\alpha^{j}_{-1},d^{k}_{-1}\right]_{+}\mid 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$

One can then impose:

$$\nu_i^{(1)} = \frac{-1}{\left(2\pi\alpha'\right)^2}\mu_i^{(1)} \tag{53}$$

to restore the value of the mass for the first excited state (for example), and in general:

$$\nu_i^{(m)} = \frac{-m^2}{\left(2\pi\alpha'\right)^2}\mu_i^{(m)} \tag{54}$$

equivalent to:

$$T_{(m)}^{ij} = \frac{-m^2}{\left(2\pi\alpha'\right)^2} D_{(m)}^{ij} \tag{55}$$

to restore those of the other levels, where $m > 0$ represents the number of state level. Finally, we obtain (Table 4).

Table 4.
This table represents the first levels of the mass spectrum after GSO projection and the application of the equation (54).

Level	**N-S Sector**		**R-Sector**	
	State	**Mass**	**State**	**Mass**
1	$b^i_{-\frac{1}{2}}\|0\rangle$	0	$\|0\rangle$	0
3	$\left\langle b^i_{-\frac{1}{2}}, b^j_{-\frac{1}{2}}, b^k_{-\frac{1}{2}} \right\rangle_- \|0\rangle$	$\frac{1}{\alpha'}$	$d^j_{-1}\|0\rangle$	$\frac{1}{\alpha'}$
	$b^i_{-\frac{3}{2}}\|0\rangle$ $\left[U_1^{-1}\alpha^i_{-1}, b^j_{-\frac{1}{2}}\right]_+ \|0\rangle$	$\frac{1}{\alpha'}$ $\frac{1}{\alpha'}$	$U_1^{-1}\alpha^i_{-1}\|0\rangle$	$\frac{1}{\alpha'}$
5	$\left\langle b^i_{-\frac{1}{2}}, b^j_{-\frac{1}{2}}, b^k_{-\frac{1}{2}}, b^l_{-\frac{1}{2}}, b^m_{-\frac{1}{2}} \right\rangle_- \|0\rangle$	$\frac{2}{\alpha'}$	$d^j_{-2}\|0\rangle$	$\frac{2}{\alpha'}$
	$b^i_{-\frac{5}{2}}\|0\rangle$ $\frac{1}{3!} < b^i_{-\frac{1}{2}}, b^j_{-\frac{1}{2}}, b^k_{-\frac{1}{2}} >_-$ $\frac{1}{2!}\left[U_2^{-1}\alpha^j_{-2}, b^k_{-\frac{1}{2}}\right]_+ \|0\rangle$	$\frac{2}{\alpha'}$ $\frac{2}{\alpha'}$ $\frac{2}{\alpha'}$	$\left[d^j_{-1}, d^k_{-1}\right]_- \|0\rangle$ $U_2^{-1}\alpha^j_{-2}\|0\rangle$	$\frac{2}{\alpha'}$ $\frac{2}{\alpha'}$
	$\left\langle U_1^{-1}\alpha^j_{-1}, U_1^{-1}\alpha^k_{-1}, b^l_{-\frac{1}{2}} \right\rangle_+ \|0\rangle$	$\frac{2}{\alpha'}$	$\left[\left(U_1^{-1}\alpha^j_{-1}\right), \left(U_1^{-1}\alpha^k_{-1}\right)\right]_+ \|0\rangle$	$\frac{2}{\alpha'}$
	$\left\langle U_1^{-1}\alpha^j_{-1}, b^k_{-\frac{1}{2}}, b^l_{-\frac{1}{2}}, b^m_{-\frac{1}{2}} \right\rangle_+ \|0\rangle$ $\left[U_1^{-1}\alpha^j_{-1}, b^k_{-\frac{3}{2}}\right]_+ \|0\rangle$	$\frac{2}{\alpha'}$ $\frac{2}{\alpha'}$	$\left[\left(U_1^{-1}\alpha^j_{-1}\right), d^k_{-1}\right]_+ \|0\rangle$	$\frac{2}{\alpha'}$

By applying ($U_m U_m^{-1}$) on the both sides of (50) and (51), one can show that the equation (55) can be expressed with respect to θ_m and γ_m.

$$\gamma_{(m)}^{ij} = \frac{-m^2}{\left(2\pi\alpha'\right)^2}\theta_{(m)}^{ij} \tag{56}$$

From this result, we can fix our starting model (2) by imposing to $\theta^{\mu\nu}$ and $\gamma^{\mu\nu}$ the following relation:

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$$\gamma_{(m)}^{\mu\nu} = \frac{-m^2}{\left(2\pi\alpha'\right)^2}\theta_{(m)}^{\mu\nu} \tag{57}$$

where $m \neq 0$ and $\mu,\nu = 0,1...,D-1$.

With this condition (57), one can easily verify that all the anomaly terms (15), (19), (20), (24) and (25) of the modified Virasoro algebra due to the noncommutativity are eliminated. This result is a direct consequence of the fact that we considered noncommutativity between coordinates and moments instead of only between coordinates.

In the other hand, the Lorentz algebra's anomaly term (30) is simplified to:

$$\begin{aligned}
T^{\mu\nu\rho\lambda} &= i\pi^2\begin{pmatrix}\gamma_0^{\nu\lambda}x^\mu x^\rho + \gamma_0^{\nu\rho}x^\mu x^\lambda \\ +\gamma_0^{\mu\lambda}x^\nu x^\rho + \gamma_0^{\mu\rho}x^\nu x^\lambda\end{pmatrix} \\
&+2i\pi^2\alpha'\tau\begin{pmatrix}\gamma_0^{\nu\rho}x^\mu p^\lambda - \gamma_0^{\mu\lambda}x^\rho p^\nu \\ +\gamma_0^{\nu\lambda}x^\mu p^\rho - \gamma_0^{\mu\rho}x^\lambda p^\nu \\ +\gamma_0^{\mu\rho}x^\nu p^\lambda - \gamma_0^{\nu\lambda}x^\rho p^\mu \\ +\gamma_0^{\mu\lambda}x^\nu p^\rho - \gamma_0^{\nu\rho}x^\lambda p^\mu\end{pmatrix} + \\
&\left(i\theta_0^{\mu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\rho}\right)p^\lambda p^\nu \\
&+\left(i\theta_0^{\mu\lambda} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\lambda}\right)p^\rho p^\nu \\
&+\left(i\theta_0^{\nu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\rho}\right)p^\lambda p^\mu \\
&+\left(i\theta_0^{\nu\lambda} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\lambda}\right)p^\rho p^\mu
\end{aligned} \tag{58}$$

For the zero mode noncommutativity parameters, if we impose that $\theta_0^{\mu\nu} = \gamma_0^{\mu\nu} = 0$, the Lorentz algebra is restored where (27), (28) and (29) become as the ordinary ones, despite of the fact that the noncommutativity is still present in the relations (2) and (8).

8. Summary and Results

To conclude, we have investigated the free open fermionic string theory in a noncommutative target phase-space. We postulated the noncommutation relations (2) and derived the ones in the paraquantum case (34). We found that the modification in the commutation relations in terms of oscillating modes introduce a new anomaly terms in the Neuveu-Schwarz and Ramond Virasoro super-algebras. The Lorentz covariance is affected by the noncommutativity and the mass operator becomes non-diagonal in the usual Fock space. A redefinition of this latter is possible and a diagonalized mass operator is obtained. We then imposed specific constraints on the noncommutativity parameters $\theta^{\mu\nu}$ and $\gamma^{\mu\nu}$ to cancel the anomaly terms and recover the standard mass spectrum. Under these conditions, the GSO projection becomes applicable, allowing for the restoration of spacetime supersymmetry. Finally, to recover Lorentz invariance, we required the vanishing of the zero modes of the noncommutativity parameters, namely $\gamma_0^{\mu\nu} = 0$ and $\theta_0^{\mu\nu} = 0$. As a result, equations (27), (28), and (29) reduce to their standard forms, while noncommutativity still remains encoded in equations (2) and (8). Overall, our analysis shows complete consistency between the results obtained from standard quantization and those from the paraquantum approach.

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Appendix A.

As previously discussed, we now compute part of the mass spectrum in the paraquantized framework. Specifically, for the Ramond sector, we present the calculation of the second excited state as a representative example. The corresponding mass operator is given in equation (48),

$$M_R^2 = \frac{1}{2\alpha'}\left(\sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left[\alpha^i_{-n},\alpha^i_n\right]_+ + \sum_{i=2}^{D-1}\sum_{r=1}^{\infty} r\left[d^i_{-r},d^i_r\right]_-\right) \tag{59}$$

and by using (35):

$$\left[\alpha^i_m,\left[\alpha^j_n,\alpha^k_l\right]_+\right] = 2\left\{\begin{matrix}\left(m\eta^{ij} + i\frac{n^2}{2\alpha'}\theta^{ij}_n + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma^{ij}_n\right)\delta_{m+n}\alpha^k_l + \\ \left(m\eta^{ik} + \frac{n^2}{2\alpha'}\theta^{ik}_l + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma^{ik}_l\right)\delta_{m+l}\alpha^j_n\end{matrix}\right\} \tag{60}$$

$$\left[d^i_n,\left[d^j_m,d^k_l\right]_-\right] = 2\left(\delta^{ij}\delta_{n+m}d^k_l - \delta^{ik}\delta_{n+l}d^j_m\right) \tag{61}$$

we get:

$$M_R^2\frac{1}{2!}\left[U_1^{-1}\alpha^j_{-1},d^k_{-1}\right]_+|0\rangle = \frac{1}{2\alpha'}\left(\sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left[\alpha^i_{-n},\alpha^i_n\right]_+ + \sum_{i=2}^{D-1}\sum_{r=1}^{\infty} r\left[d^i_{-r},d^i_r\right]_-\right)\left[U_1^{-1}\alpha^j_{-1},d^k_{-1}\right]_+|0\rangle \tag{62}$$

We start with the first part of the right-hand side of the equation (62):

$$\sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left[\alpha^i_{-n},\alpha^i_n\right]_+\left[U_1^{-1}\alpha^j_{-1},d^k_{-1}\right]_+|0\rangle = \sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left[\alpha^i_{-n},\alpha^i_n\right]_+\left(U_1^{-1}\alpha^j_{-1}d^k_{-1} + d^k_{-1}U_1^{-1}\alpha^j_{-1}\right)|0\rangle \tag{63}$$

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$$= \sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left\{\begin{array}{l}\underbrace{U_1^{-1}\alpha_{-1}^{j}\left[\alpha_{-n}^{i},\alpha_{n}^{i}\right]_+ d_{-1}^{k} - 2\left\{\begin{array}{l}\left(\begin{array}{l}-\eta^{ji}U_1^{-1}+\frac{1}{2\alpha'}U_1^{-1}i\theta_n^{ji}+\\ \frac{\left(2\pi\alpha'\right)^2}{2\alpha'}U_1^{-1}i\gamma_n^{ji}\end{array}\right)\delta_{-1-n}\alpha_n^{i}+\\ \left(\begin{array}{l}-\eta^{ji}U_1^{-1}+\frac{1}{2\alpha'}U_1^{-1}i\theta_n^{ji}+\\ \frac{\left(2\pi\alpha'\right)^2}{2\alpha'}U_1^{-1}i\gamma_n^{ji}\end{array}\right)\delta_{-1+n}\alpha_{-n}^{i}\end{array}\right\}d_{-1}^{k}\,|0\rangle}_{\sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left[\alpha_{-n}^{i},\alpha_{n}^{i}\right]_+ U_1^{-1}\alpha_{-1}^{j}d_{-1}^{k}|0\rangle}\\ \underbrace{+d_{-1}^{k}U_1^{-1}\alpha_{-1}^{j}\left[\alpha_{-n}^{i},\alpha_{n}^{i}\right]_+ - 2d_{-1}^{k}\left\{\begin{array}{l}\left(\begin{array}{l}-\eta^{ji}U_1^{-1}+\frac{1}{2\alpha'}U_1^{-1}i\theta_n^{ji}+\\ \frac{\left(2\pi\alpha'\right)^2}{2\alpha'}U_1^{-1}i\gamma_n^{ji}\end{array}\right)\delta_{-1-n}\alpha_n^{i}+\\ \left(\begin{array}{l}-\eta^{ji}U_1^{-1}+\frac{1}{2\alpha'}U_1^{-1}i\theta_n^{ji}+\\ \frac{\left(2\pi\alpha'\right)^2}{2\alpha'}U_1^{-1}i\gamma_n^{ji}\end{array}\right)\delta_{-1+n}\alpha_{-n}^{i}\end{array}\right\}|0\rangle}_{\sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\left[\alpha_{-n}^{i},\alpha_{n}^{i}\right]_+ d_{-1}^{k}U_1^{-1}\alpha_{-1}^{j}|0\rangle}\end{array}\right\} \quad (64)$$

and for the second term of the right-hand side of the equation (62):

$$\sum_{i=2}^{D-1}\sum_{r=1}^{\infty} r\left[d_{-r}^{i},d_{r}^{i}\right]_-\left[U_1^{-1}\alpha_{-1}^{j},d_{-1}^{k}\right]_+|0\rangle = 4\eta^{ji}\left[U_1^{-1}\alpha_{-1}^{j},d_{-1}^{k}\right]_+ \quad (65)$$

Combining (64) and (65), one can get the final result:

$$M_R^2\frac{1}{2!}\left[U_{-1}^{-1}\alpha_{-1}^{j},d_{-1}^{k}\right]_+|0\rangle = \frac{1}{4\alpha'}4\left(1-\frac{1}{2\alpha'}\mu_i^1\delta^{ji}-\frac{\left(2\pi\alpha'\right)^2}{2\alpha'}\nu_i^1\delta^{ji}+1\right)\left[U_{-1}^{-1}\alpha_{-1}^{j},d_{-1}^{k}\right]_+|0\rangle \quad (66)$$

So,

$$M_R^2\frac{1}{2!}\left[U_{-1}^{-1}\alpha_{-1}^{j},d_{-1}^{k}\right]_+|0\rangle = \frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}-\left(2\pi\alpha'\right)^2\nu_j^{(1)}\right)\right)\left[U_{-1}^{-1}\alpha_{-1}^{j},d_{-1}^{k}\right]_+|0\rangle \quad (67)$$